GCSE Maths – Geometry and Measures

Circle Theorems (Higher Only)

Notes

WORKSHEET

Circle Theorems

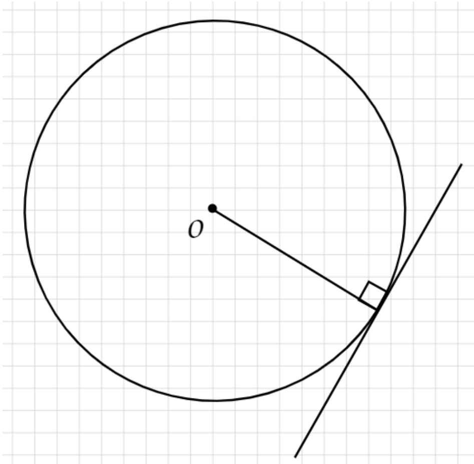
A **theorem** is a statement which can be proven to be true. **Circle theorems** involve properties of circles.

You will need to be able to **identify**, **use** and **prove** seven circle theorems. We will go through each one of them in detail. The order of the following theorems does not matter.

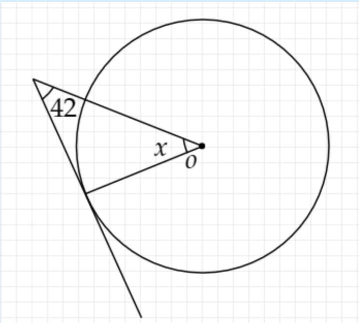
Theorem 1

When a radius meets a tangent, the angle will always be 90°

- A **tangent** is a straight line which touches the circle once at a single point on the circumference of the circle. The position of the **tangent** does not matter, it can be anywhere around the circle.
- Point o in the diagram is the centre of the circle.
- The line connecting the centre of the circle and the **tangent** is the **radius** of the circle.
- Where the radius and the **tangent** meet, they make a right-angle.



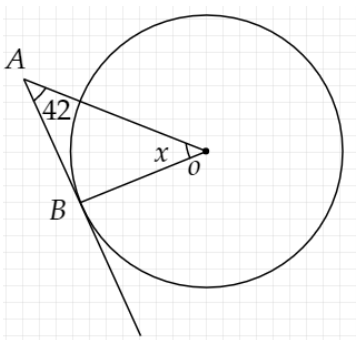
Example: In the following diagram, find the value of x.
Diagram not drawn to scale.



1. Label the unlabelled points.

OB: radius of the circle
AB: tangent to the circle

*Using **theorem 1**: Angle $OBA = 90°$*
since the radius meets the tangent at that point.



2. Use the property of angles in a triangle to find angle x.

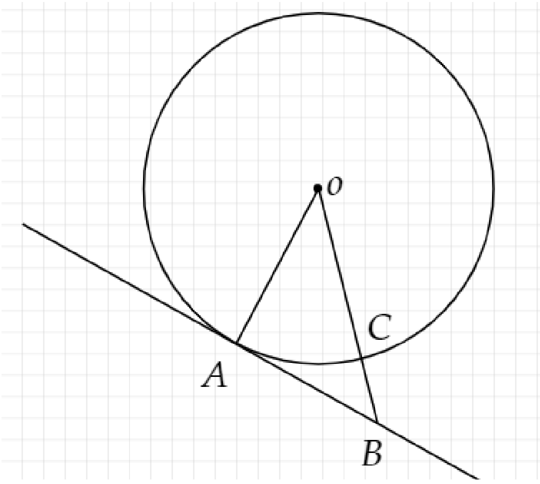
Sum of all angles inside a triangle sum to 180°:

$$42° + 90° + x = 180°$$
$$x = 180° - 90° - 42° = \mathbf{48°}$$

Proof of Theorem 1

There is **no proof** that you need to remember for this theorem because it comes directly from the **definition of a tangent**. The definition of the **tangent** is that it is **perpendicular** to the radius. Therefore, if the line the radius meets is specified as a **tangent**, the angle between them will be $90°$ because they are perpendicular to each other.

However, below is a proof that explains why the angle is $90°$.
We will use the following diagram:



STEP 1: Make the necessary assumptions which are opposite to what you are trying to prove.

OAB is a right-angled triangle.

So, if OA is not perpendicular to AB, then we will assume OB is perpendicular to AB.

Therefore, angle $\boldsymbol{OBA} = \mathbf{90°}$

STEP 2: Progress with the assumptions and prove that they are wrong and impossible to be true.

If angle $OBA = 90°$:

OA should be the longest side because it is opposite the biggest angle.

This makes OA the **hypotenuse**.

Therefore, $\boldsymbol{OA} > \boldsymbol{OB}$.

However, $\boldsymbol{OA} = \boldsymbol{OC}$ because both are radii of the circle, therefore, $\boldsymbol{OC} > \boldsymbol{OB}$ as well. But,

$$\boldsymbol{OB} = \boldsymbol{OC} + \boldsymbol{CB}$$

This means OC cannot be greater than OB and as $OA = OC$, OA cannot be greater than OB.

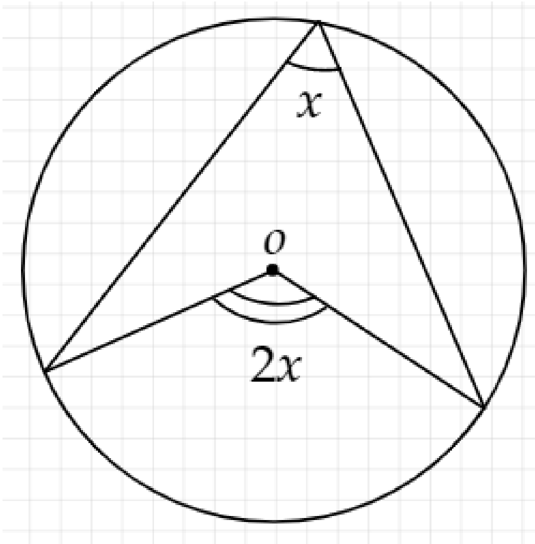
Therefore, OB is the **hypotenuse** which means $\boldsymbol{OAB} = \mathbf{90°}$.
This proves our assumption that angle $\boldsymbol{OBA} = \mathbf{90°}$ was wrong.
Hence, for OAB to be a right angled triangle we must have angle $\boldsymbol{OAB} = \mathbf{90°}$.

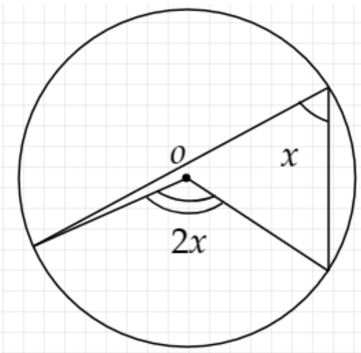
Theorem 2

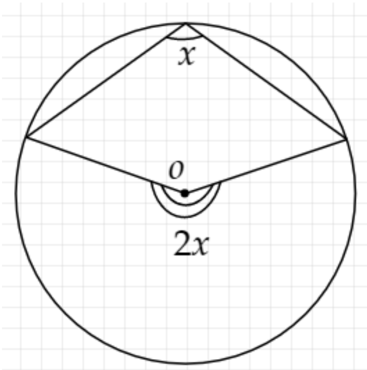
Angle at the centre of the circle is twice the angle at the circumference

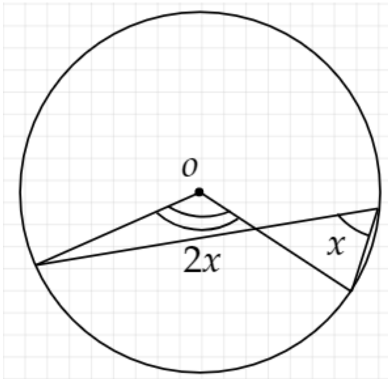
- Point o in the diagram is the centre of the circle.
- It is important that angle x is at the **circumference** of the circle.
- An **isosceles** triangle can be made from the two radii of the circle.

- The representation on the right is not the only representation possible. The following are other ways the theorem can be applied.

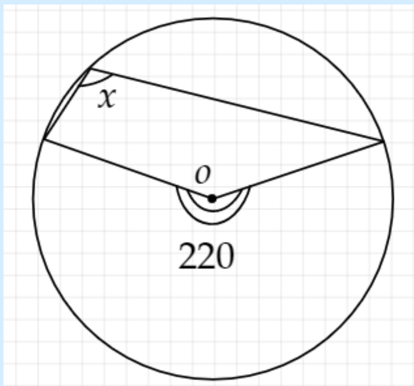








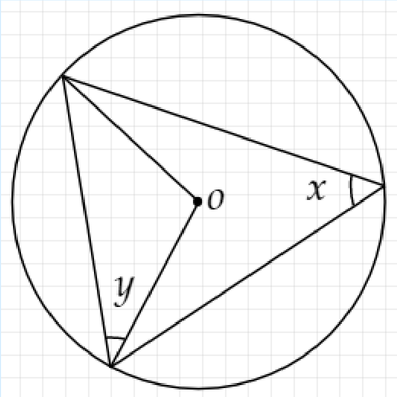
Example: In the following diagram, find the value of x.
Diagram not drawn to scale.



*Using **theorem 2,** the angle at the centre is twice the angle at the circumference, so:*

$$x = \frac{220°}{2} = \mathbf{110°}$$

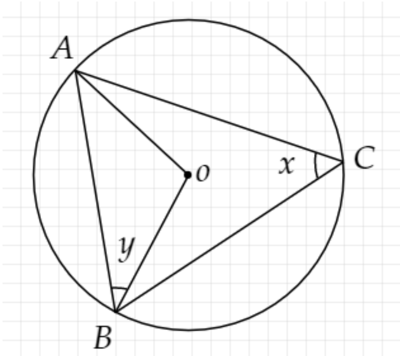
Example: In the following diagram, find the value of x if $y = 35$.
Diagram not drawn to scale.



This is a more challenging question. If you spot the isosceles triangle, then you can easily work your way to the answer.

1. Label all the unlabelled points.

To find angle x, we will first need to find angle AOB at the centre.



2. Identify the isosceles triangle within the circle.

OA = *radius of the circle*
OB = *radius of the circle*

Therefore, $OA = OB$ and *hence triangle* OAB *is isosceles.*

3. Use the property of isosceles triangles to find the angle at the centre.

Property of isosceles triangle is that base angles are equal:
Therefore, $\angle OAB = \angle OBA = y = 35°$

Using sum of all angles inside a triangle sum to $180°$*:*

$$35° + 35° + \angle BOA = 180°$$
$$\angle BOA = 180° - 35° - 35° = 110°$$

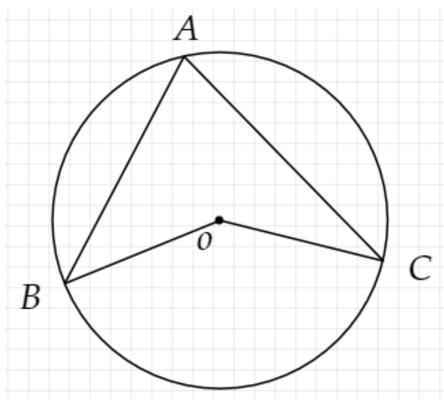
4. Use theorem 2 to find x.

*Using **theorem 2**, angle at the centre is twice the angle at the circumference:*

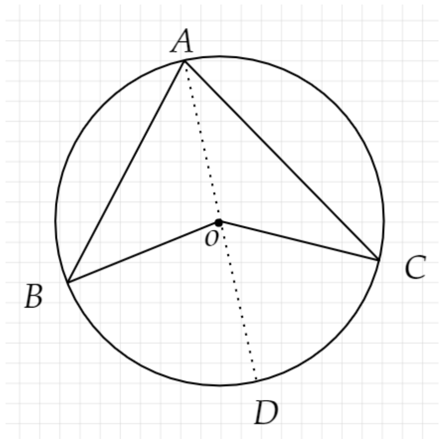
$$x = \frac{110°}{2} = \mathbf{55°}$$

Proof of Theorem 2

We will start with the following labelled diagram to achieve a proof for the theorem. For the following diagram, we will prove that $\angle \boldsymbol{BOC} = \mathbf{2} \times \angle \boldsymbol{BAC}$.



STEP 1: Draw a diameter from point A to a point D.



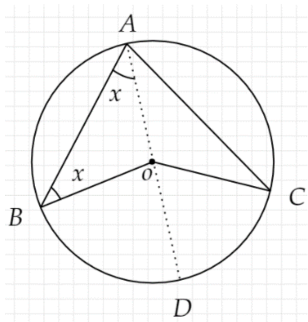
Notice: $OA = OB = OC$
since all of them are radii of the circle.

Therefore, triangle AOB and triangle AOC are **isosceles**.

Property of **isosceles** triangles: base angles are equal.

STEP 2: Starting from one of the **isosceles** triangles, label all angles using a single variable. The do the same with the other **isosceles** triangle with a different variable.

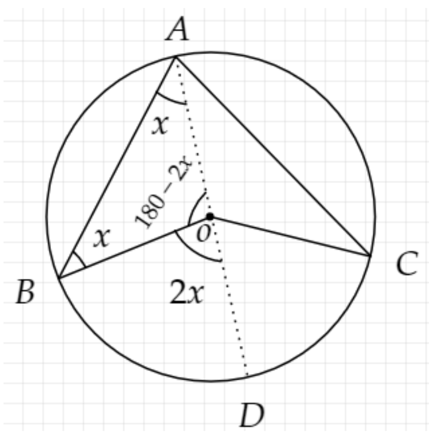
Triangle $\boldsymbol{AOB}$: Label the equal base angles as x.



All angles inside a triangle add to $180°$:

$$x + x + \angle AOB = 180°$$
$$\angle AOB = 180° - 2x$$

Angles on a straight line add to $180°$.
The diameter is a straight line so the above rule applies:



$$\angle AOB + \angle DOB = 180°$$
$$180° - 2x + \angle DOB = 180°$$

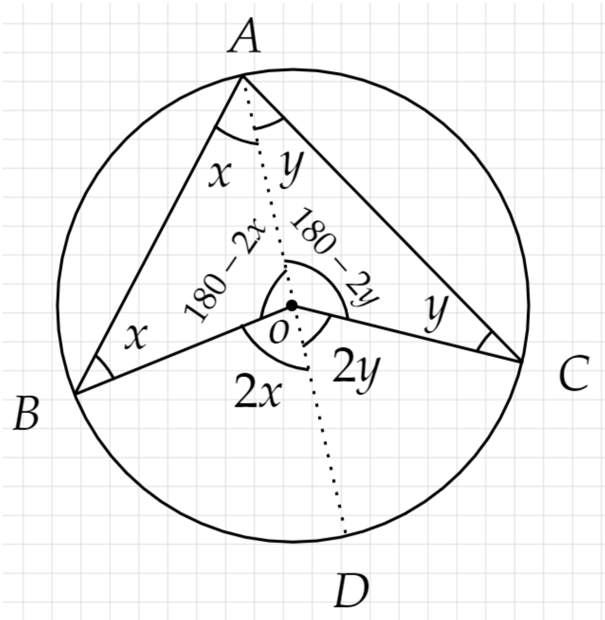
$$\angle DOB = 2x$$

Similarly, labelling the other isosceles triangle AOC:

$$\angle BOC = 2x + 2y = 2(x + y)$$
$$\angle BAC = x + y$$

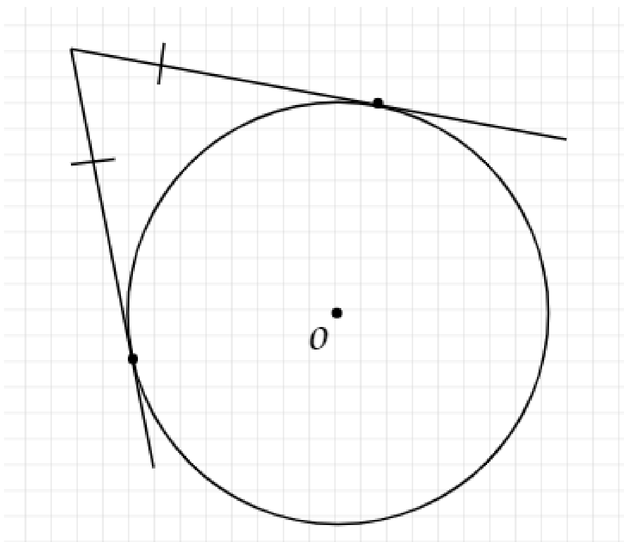
Therefore, $\boldsymbol{\angle BOC = 2 \times \angle BAC}$.

Hence, we have proven the theorem that an angle at the centre of the circle is twice the angle at the circumference.



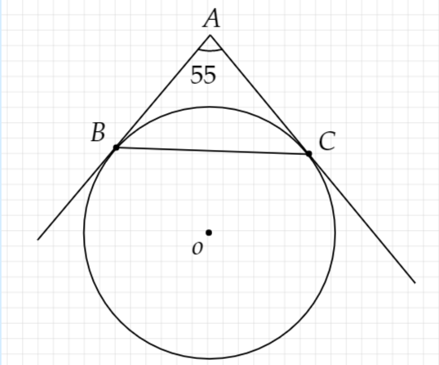
Theorem 3

Two tangents from a single point are equal in length from that point to where they touch the circumference of the circle.



- The two **tangents** are only equal till the point where they touch the circle.
- An **isosceles** triangle can be made from these two tangents since they each have the **same length** measured from the **point of intersection** to the **points of tangency**.

Example: In the following diagram, find the value of angle ABC. Diagram not drawn to scale.



Using theorem 3, two tangents from the same point are equal from that point to where they touch the circumference of the circle so:

$$AB = AC$$

Therefore, triangle $\boldsymbol{ABC}$ *is isosceles. Hence, by the property of isosceles triangles, the base angles are equal:*

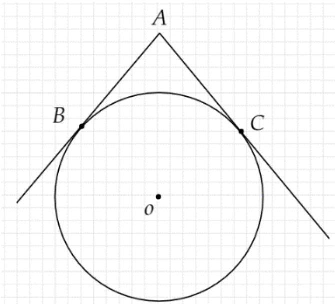
$$\angle ABC = \angle ACB = x$$

Using the property that the sum of all angles inside a triangle equal $180°$*:*

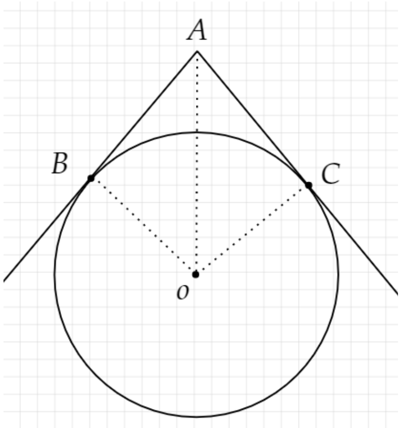
$$55° + x + x = 180°$$
$$2x = 180° - 55°$$
$$x = \frac{125°}{2} = 62.5°$$

Proof of Theorem 3

We will start with the following labelled diagram. We are proving that $AB = AC$.



STEP 1: Join all the points to the centre, forming two triangles.



STEP 2: Prove these triangles are congruent.

Since they both are radii of the circle, $\boldsymbol{OB = OC}$.

$\boldsymbol{OA}$ is the same for both the triangles.

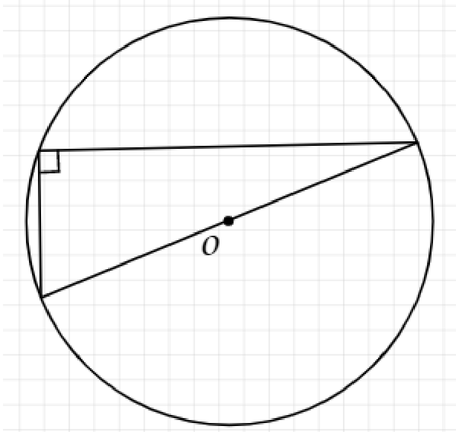
$\boldsymbol{\angle OBA = \angle OCA = 90°}$ since the radius is always perpendicular to a tangent by **theorem 1**.

Hence, triangles are **congruent** and therefore we've proven the required result that $\boldsymbol{AB = AC}$.

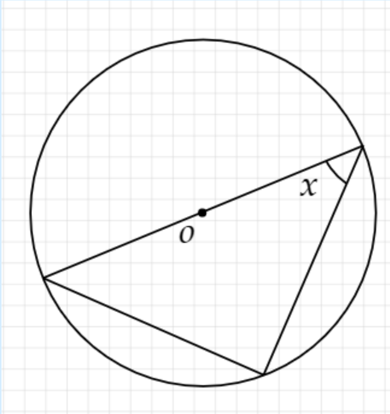
Theorem 4

Angle inside a semi-circle is always 90°

- For this theorem, the triangle can be formed in any way, however each point should **touch the circumference** of the circle and the **hypotenuse** must form a diameter of the circle.



Example: In the following diagram, one of the angles is $40°$ and another of the angles is labelled x, as shown in the diagram. Given that $x \neq 40°$, find the value of x.
Diagram not drawn to scale.



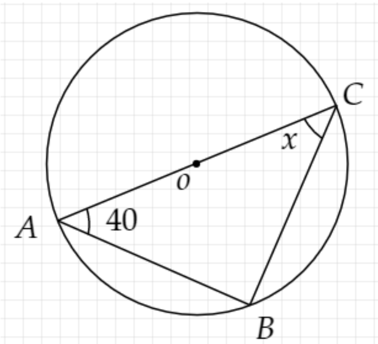
1. Label all the points on the circle.

2. Use theorem 4 to find a missing angle.

   *Using **theorem 4,** since the angle inside a semicircle is always* $90°$:

   $$\angle \boldsymbol{CBA} = \mathbf{90°}$$

   This means that we must have $\angle CAB = 40°$.



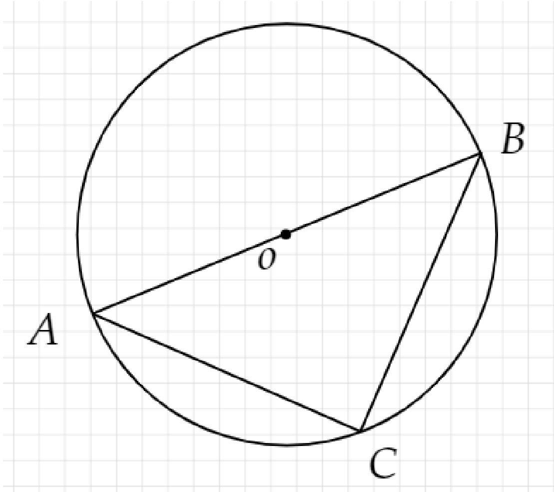
3. Use the property of angles in a triangle to find the missing angle.

   Using sum of all angles inside a triangle equals $180°$*:*

   $$40° + 90° + x = 180°$$
   $$x = 180° - 90° - 40° = \mathbf{50°}$$

Proof of Theorem 4

This proof uses theorem 2, as explained below.



Line AOB is a straight line, therefore

$$\angle AOB = 180°$$

Using theorem 2, the angle at the circumference is twice the angle at the centre so

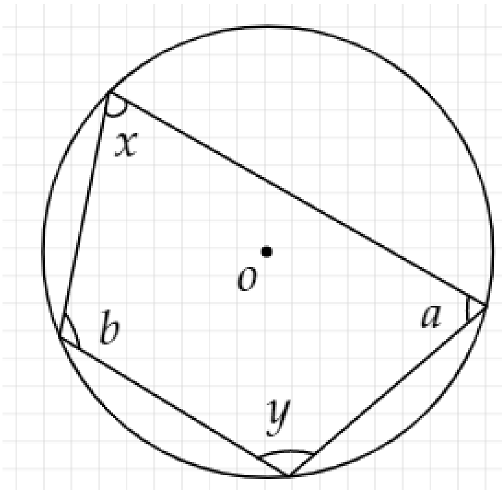
$$\angle ACB = \angle AOB \div 2$$

Therefore, $\angle ACB = 90°$.

Hence, proved.

Theorem 5

In a cyclic quadrilateral, opposite angles add up to 180°

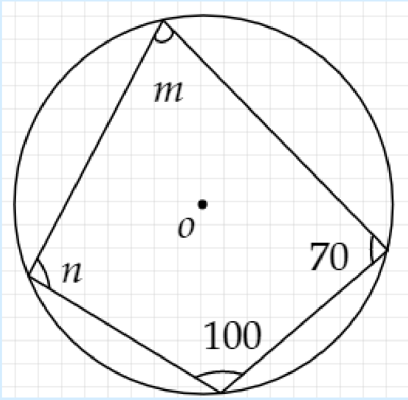


- In the above diagram,

$$x + y = 180°$$
$$a + b = 180°$$

- It is important to note that to make a **cyclic quadrilateral**, all vertices should be touching the circumference of the circle.

Example: In the following diagram, find the value of m and n.
Diagram not drawn to scale.



*Using **theorem 5,** opposite angles in a cyclic quadrilateral add up to* $180°$ *so:*

$$m + 100 = 180°$$
$$n + 70 = 180°$$

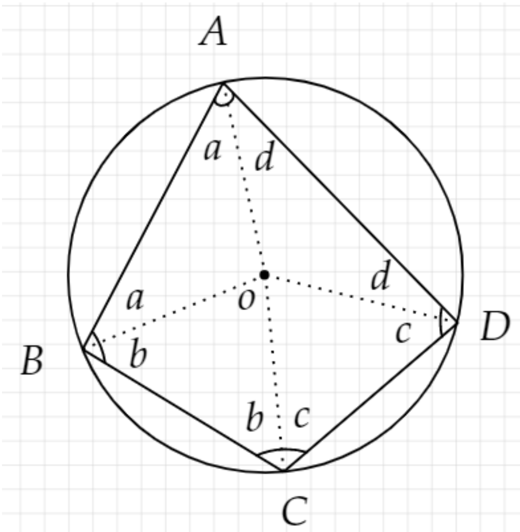
Solving these equations:

$$m = 180° - 100° = 80°$$
$$n = 180° - 70° = 110°$$

$$\boldsymbol{m = 80°}$$
$$\boldsymbol{n = 110°}$$

Proof of Theorem 5

Divide the quadrilateral into 4 **isosceles** triangles by joining all the points to the centre of the circle. As the base angles of an **isosceles** triangle are equal, label all the base angles using different variables as shown in the diagram below.



Interior angles of a quadrilateral add to $360°$.
Therefore,

$$2a + 2b + 2c + 2d = 360°$$

Dividing 2 from both sides:

$$a + b + c + d = 180°$$

$$\angle ABC = a + b$$
$$\angle ADC = d + c$$

Therefore,

$$\boldsymbol{\angle ABC + \angle ADC = 180}$$

Similarly,

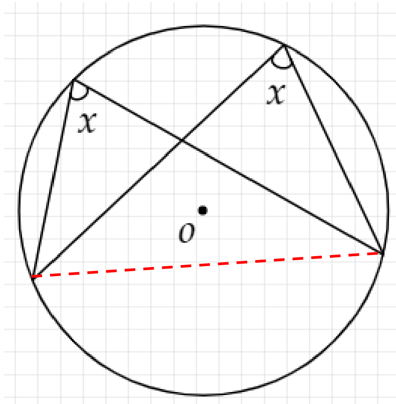
$$\boldsymbol{\angle BAD + \angle DCB = 180}$$

This shows opposite angles add to $180°$. Hence, proved.

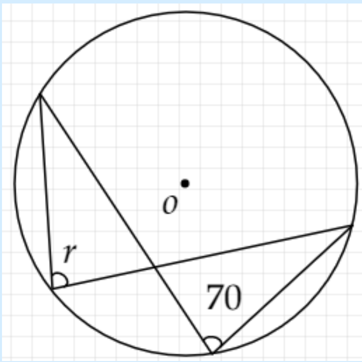
Theorem 6

Angles in the same segment are equal

- A **segment** of a circle is the region that is bounded by an arc (curve) and a chord of the circle. The red dashed line marks the chord of the **segment**. Both the angles are in the **same segment**, therefore are **equal**.
- It is important for all the points to be touching the circumference of the circle.
- It does not have to be only two angles; it can be any number of angles. However, they all must be in the same **segment**.



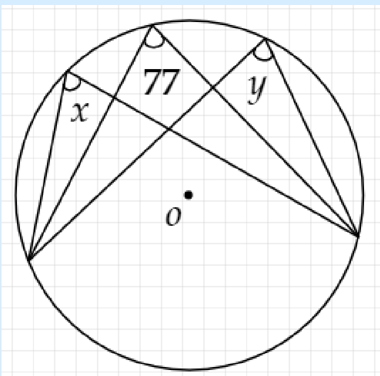
Example: In the following diagram, find the value of r. Diagram not drawn to scale.



Using theorem 6, angle r and angle $70°$ are in the same segment so since angles in the same segment are equal, we have

$$\boldsymbol{r = 70°}.$$

Example: In the following diagram, find the values of x and y. Diagram not drawn to scale.

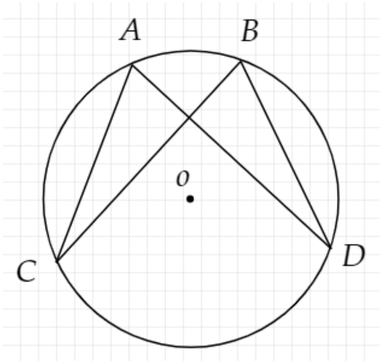


Using theorem 6, angles x, angle y and angle $77°$ all lie in the same segment so since all angles in the same segment are equal, we have

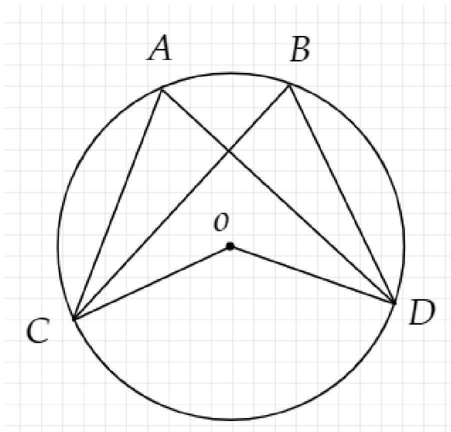
$$\boldsymbol{x = y = 77°}$$

Proof of Theorem 6

This proof uses theorem 2 as shown below. We are trying to prove that $\angle CAD = \angle CBD$.



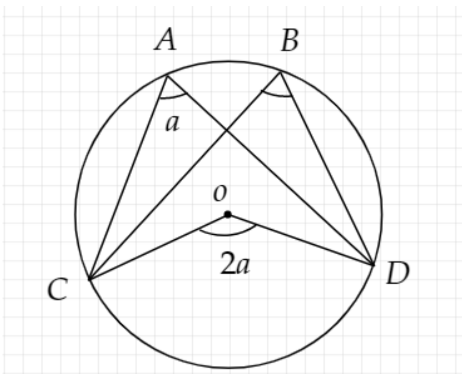
STEP 1: Make an angle at the centre of the circle, joining points C and D to the centre of the circle.



STEP 2: Use theorem 2 to prove that $\angle CAD = \angle CBD$.

Let $\angle CAD = a$. We want to show that $\angle CBD = a$ as well.

If $\angle CAD = a$ then $\angle COD = 2a$ since by **theorem 2** the angle at the centre is twice the angle at the circumference.



If $\angle COD = 2a$ then $\angle CBD = a$ since by **theorem 2** the angle at the circumference is half the angle at the centre.

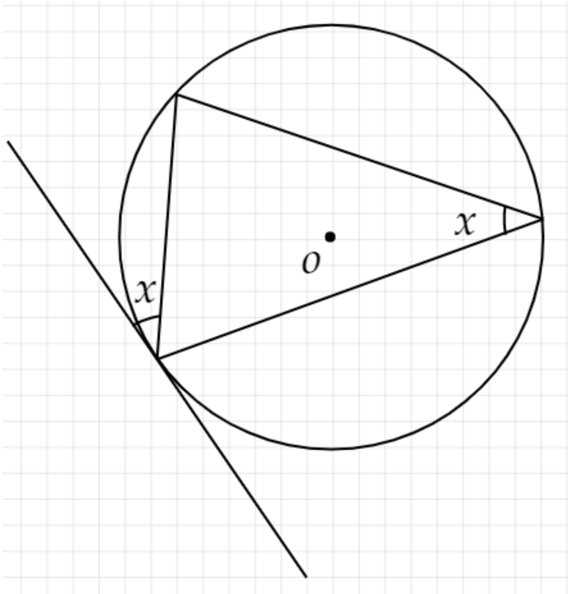
Therefore,

$$\angle CAD = \angle CBD = a$$

Hence, proved.

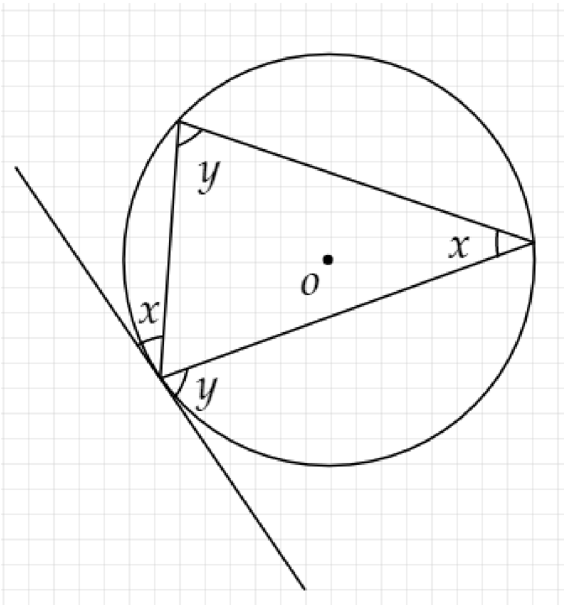
Theorem 7

The angle between a chord and a tangent is equal to the angle in the alternate segment



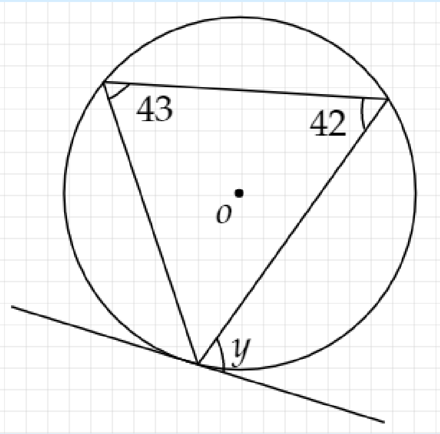
- This is known as the **alternate segment theorem**.

- From the above diagram, the following would also be true:



- The angle on the tangent is not inside the circle, it is from the chord to the tangent.
- The points must touch the circumference of the circle for these observations to be true.

Example: In the following diagram, find the value of y. Diagram not drawn to scale.

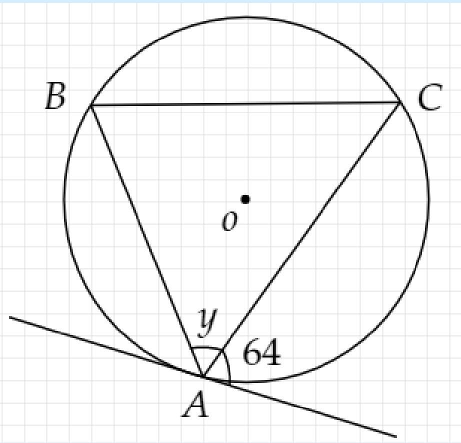


Using theorem 7 (alternate segment theorem), the angle between a chord and a tangent is equal to the angle in the alternate segment so we must have

$$y = 43°$$

Note, $y \neq 42°$ *since these angles are in the same segment.*

Example: In the following diagram, $AB = BC$. Find the value of y. Diagram not drawn to scale.



As $\boldsymbol{AB} = \boldsymbol{BC}$*, triangle* $\boldsymbol{ABC}$ *is an isosceles triangle. Therefore, base angles are equal.*

Hence,

$$\boldsymbol{\angle ABC = \angle ACB}$$

Using theorem 7,

$$\boldsymbol{\angle ABC = 64°}$$

since $\angle ABC$ *is in the alternate segment to the angle measuring* $64°$*.*

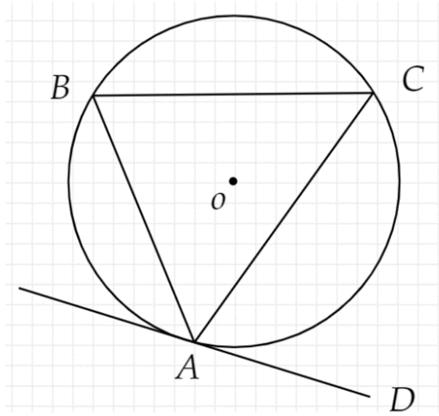
Using the property that the sum of all angles inside a triangle equal $180°$*:*

$$64° + 64° + y = 180°$$

$$y = 180° - 64° - 64° = \mathbf{52°}$$

Proof of Theorem 7

This is a challenging proof. Let's start with the following diagram.
We are proving that $\angle ABC = \angle CAD$.



STEP 1: Make a diameter through A, labelling the end point E and joining it to C. Then try labelling as many angles as possible using a single variable.

Let $\boldsymbol{\angle CAD = x}$.

Then

$$\boldsymbol{\angle EAD = 90^\circ}$$

since tangent meets radius at 90°.

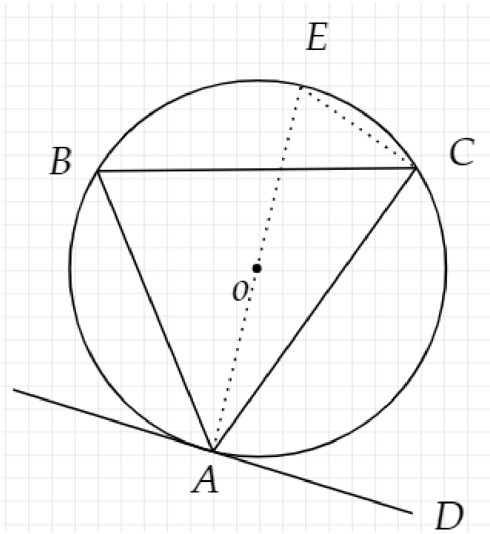
Therefore,

$$\boldsymbol{\angle EAC = 90^\circ - x}$$

Now,

$$\boldsymbol{\angle ACE = 90^\circ}$$

since by **theorem 4** angle inside a semicircle on the circumference is always right angled.



STEP 2: Use properties of triangles and theorem 6 to prove that $\angle ABC = \angle CAD$.

Angles in a triangle add to 180°:

$$(90^\circ - x) + 90^\circ + \angle CEA = 180^\circ$$
$$\boldsymbol{\angle CEA = x}$$

Then

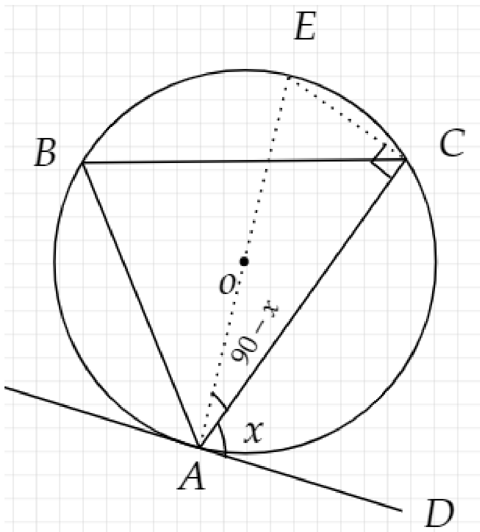
$$\boldsymbol{\angle ABC = \angle CEA = x}$$

since by **theorem 6** angles in the same segment are equal.

Therefore,

$$\boldsymbol{\angle ABC = \angle CAD = x}$$

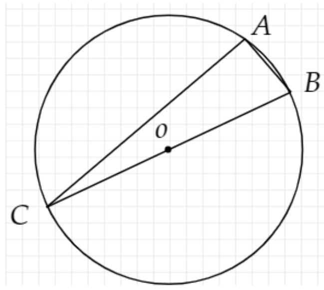
Hence, proved.



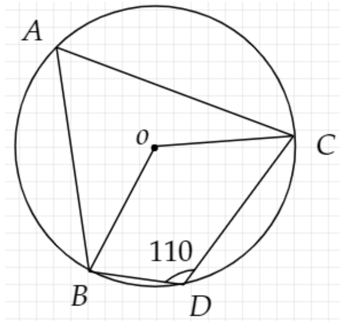
Circle Theorems – Practice Questions

The following diagrams are not drawn to scale.

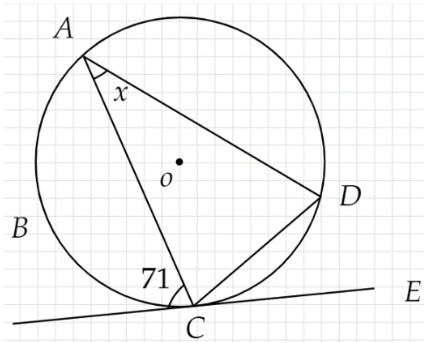
1. If $\angle ACB = 33°$, work out the value of $\angle ABC$.



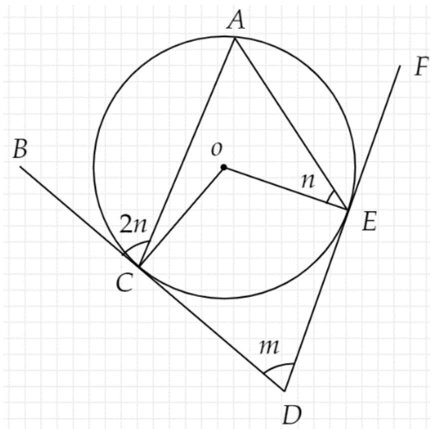
2. Find $\angle BOC$.



3. If $\angle ACD = 65°$, find x.



4. Prove that $m = 2n$.



Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.